LARGE DEVIATIONS PRINCIPLE FOR THE MEAN-FIELD HEISENBERG MODEL WITH EXTERNAL MAGNETIC FIELD

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Abstract: In this paper, we consider the mean-field Heisenberg model with deterministic external magnetic field. We prove a large deviation principle for S_n/n with respect to the associated Gibbs measure, where S_n/n is the scaled partial sum of spins. In particular, we obtain an explicit expression for the rate function.

1 Introduction

The Ising model and the Heisenberg model are two main statistical mechanical models of ferromagnetism. The Ising model is simpler and better understood. The limit theorems for the total spin in the mean-field Ising model (also called the Curie-Weiss model) were shown by Ellis and Newman [4]. Recently, it was shown by Chatterjee and Shao [1], and independently by Eichelsbacher and M. Löwe [3], that the total spin satisfies a Berry-Esseen type error bound of order $1/\sqrt{n}$ at both the critical temperature and non-critical temperature.

The Heisenberg model is more realistic and more challenging. There are few results on limit theorems known for this model. Recently, Kirkpatrick and Meckes [6] proved a large deviation principle and central limit theorems for the total spin in mean-field Heisenberg model without deterministic external magnetic field. The Berry-Esseen bound for the total spin in a more general model (i.e., the mean-field $O(N)$ model) with optimal bounds was obtained in [7] by using Stein's method. In this paper, we consider the mean-field Heisenberg model with deterministic external magnetic field. We prove a large deviation principle for the total spin with respect to the associated Gibbs measure. In particular, we obtain an explicit expression for the rate function.

Let $\mathbb{S}^2$ denote the unit sphere in $\mathbb{R}^3$. In this paper, we consider the mean-field Heisenberg model, where each spin σ_i is in $\mathbb{S}^2$, at a complete graph vertex i among n vertices, $n \geq 1$.

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The state space is $\Omega_n = (\mathbb{S}^2)^n$ with product measure $\mathbb{P}_n = \mu \times \cdots \times \mu$, where μ is the uniform probability measure on $\mathbb{S}^2$. The Hamiltonian of the Heisenberg model with external magnetic field $h \in \mathbb{R}^3 \setminus (0,0,0)$ can be described by $\mathrm{H}_n(\sigma) = -\dfrac{1}{2n} \sum_{i=1}^n \sum_{j=1}^n \langle \sigma_i, \sigma_j \rangle - \langle h, \sum_{i=1}^n \sigma_i \rangle = -\dfrac{1}{2n} S_n(\sigma)^2 - \langle h, S_n(\sigma) \rangle$, (0)where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^3$, $S_n(\sigma) = \sum_{i=1}^n \sigma_i$ is the total magnetization of the model. Let $\beta > 0$ be so-called the inverse temperature. The Gibbs measure is the probability measure $\mathbb{P}_{n,\beta}$ on Ω_n with density function:

$$d\mathbb{P}_{n,\beta}(\sigma) = \frac{1}{Z_{n,\beta}} \exp\left(-\beta H_n(\sigma)\right) d\mathbb{P}_n(\sigma),$$

where $Z_{n,\beta}$ is the partition function:

$$Z_{n,\beta} = \int_{\Omega_n} \exp\left(-\beta H_n(\sigma)\right) d\mathbb{P}_n(\sigma).$$

In 2013, Kirkpatrick and Meckes [6] studied limit theorems for the mean-field Heisenberg model without external magnetic field, i.e., there is no the second term in (1.1). They proved a large deviation result for the total spin

$$S_n := S_n(\sigma) = \sum_{i=1}^n \sigma_i$$

distributed according to the Gibbs measures. In this paper, we consider the above problem but with external magnetic field, i.e., we take $h \in \mathbb{R}^3$, $h \neq (0,0,0)$ in the expression of the Hamiltonian (1.1). The rate function in our main theorem takes a different form from that of Kirkpatrick and Meckes [6]. Besides, with the appearance of h, the computation of rate function becomes more complicated.

Before stating our main result, let us recall some basic definitions on the large deviation principle.

Definition 1.1 (Rate function)**.** Let I be a function mapping the complete, separable metric $\mathcal{X}$ into $[0,\infty]$. The function I is called a rate function if I has compact level sets, i.e., for all $M < \infty$, $\{x \in \mathcal{X} : I(x) \leq M\}$ is compact.

Here and thereafter, for $A \subset \mathcal{X}$, we write $I(A) = \inf_{x \in A} I(x)$.

Definition 1.2 (Large deviation principle)**.** Let $\{(\Omega_n, \mathcal{F}_n, \mathbb{P}_n), n \geq 1\}$ be a sequence of probability spaces. Let $\mathcal{X}$ be a complete, separable metric space, and let $\{Y_n, n \geq 1\}$ be a sequence of random variables such that Y_n maps Ω_n into $\mathcal{X}$, and I a rate function on $\mathcal{X}$.

Then Y_n is said to satisfy the large deviation principle on $\mathcal{X}$ with rate function I if the following two limits hold.

(i) **Large deviation upper bound**. For any closed subset F of $\mathcal{X}$

$$\lim_{n\to\infty}\sup \frac{1}{n}\log \mathbb{P}_n\left\{Y_n \in F\right\} \le -I(F).$$

(ii) **Large deviation lower bound**. For any open subset G of $\mathcal{X}$

$$\lim_{n\to\infty}\inf \frac{1}{n}\log \mathbb{P}_n\left\{Y_n \in G\right\} \ge -I(G).$$

Throughout this paper, $\mathcal{X}$ denotes a complete, separable metric space. The unit sphere and the unit ball in $\mathbb{R}^3$ are denoted by $\mathbb{S}^2$ and $\mathbb{B}^2$, respectively. The inner product and the Euclidean norm in $\mathbb{R}^3$ are, respectively, denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. For $x \in \mathbb{R}^3$, we write $x^2 = \langle x, x\rangle$. For a function f defines on $(a,b) \subset \mathbb{R}$ with $\lim_{x\to a^+} f(x) = y_1$ and $\lim_{x\to b^-} f(x) = y_2$, we write $f(a) = y_1$ and $f(b) = y_2$.

The following result is so-called the tilted large deviation principle, see [5; p. 34] for a proof.

Proposition 1.3. *Let $\{(\Omega_n, \mathcal{F}_n, \mathbb{P}_n), n \ge 1\}$ be a sequence of probability spaces. Let $\{Y_n, n \ge 1\}$ be a sequence of random variables such that Y_n maps Ω_n into $\mathcal{X}$ satisfying the large deviation principle on $\mathcal{X}$ with rate function I. Let ψ be a bounded, continuous function mapping $\mathcal{X}$ into $\mathbb{R}$. For $A \in \mathcal{F}_n$, we define a new probability measure*

$$\mathbb{P}_{n,\psi} = \frac{1}{Z}\int_A \exp\left[-n\psi(Y_n)\right] d\mathbb{P}_n,$$

where

$$Z = \int_{\Omega_n} \exp\left[-n\psi(Y_n)\right] d\mathbb{P}_n.$$

Then with respect to $\mathbb{P}_{n,\psi}, Y_n$ satisfies the large deviation principle on $\mathcal{X}$ with rate function

$$I_\psi(x) = I(x) + \psi(x) - \inf_{y\in\mathcal{X}}\left\{I(y) + \psi(y)\right\},\ x \in \mathcal{X}.$$

Kirkpatrick and Meckes [6] used Sanov's theorem [2; p. 16] to prove the following large deviation principle for S_n/n in the absence of external magnetic field, i.e., in the expression of the Hamiltonian (1.1), letting $h = (0,0,0)$. Their result reads as follows. Note that in Theorem 5 in Kirkpatrick and Meckes [6], the author missed to indicate the case where $\beta > 3$.

Theorem 1.4. *[6; Theorem 5] Consider the mean-field Heisenberg model in the absence of external magnetic field. Let* $S_n = \sum_{i=1}^n \sigma_i$. *Then* S_n/n *satisfies a large deviation principle with respect to the Gibbs measure* $\mathbb{P}_{n,\beta}$ *with rate function*

$$I(x) = \begin{cases} a\coth(a) - 1 - \log\left(\dfrac{\sinh(a)}{a}\right) - \dfrac{\beta x^2}{2} & \text{if } \beta \leq 3, \\ a\coth(a) - 1 - \log\left(\dfrac{\sinh(a)}{a}\right) - \dfrac{\beta x^2}{2} + \log\left(\dfrac{\sinh(b)}{b}\right) - \dfrac{\beta}{2}\left(\coth(b) - \dfrac{1}{b}\right)^2 & \text{if } \beta > 3, \end{cases}$$

where a *is defined by* $\coth(a) - \dfrac{1}{a} = ||x||$, *and* b *is defined by* $\coth(b) - \dfrac{1}{b} = \dfrac{b}{\beta}$.

2 Main result

In the following, we prove a large deviation principle for the mean-field Heisenberg model with external magnetic field. The proof relies on Cramér theorem (see, e.g., [2; p. 36]) and the titled large deviation principle (Proposition 1.3). The following theorem is the main result of this paper. For all $n \geq 1$, since σ_i takes values in $\mathbb{S}^2$ for $1 \leq i \leq n$, we see that $\sum_{i=1}^n \sigma_i/n$ takes values in $\mathbb{B}^2$. Differently from Kirkpatrick and Meckes [6; Theorem 5] (Theorem 1.4 in this paper), when we consider the mean field Heisenberg model with external magnetic field, the rate function in Theorem 2.1 takes only one form for all $\beta > 0$. In Theorem 2.1 below, if $h = (0, 0, 0)$, then the rate function $I_\psi(x)$ coincides with the rate function $I(x)$ in Theorem 1.4 for the case where $\beta > 3$.

Theorem 2.1. *Consider the mean-field Heisenberg model with the Hamiltonian in [1.1]. Let* $S_n = \sum_{i=1}^n \sigma_i$. *Then* S_n/n *satisfies a large deviation principle with respect to the measure* $\mathbb{P}_{n,\beta}$ *with rate function*

$$I_\psi(x) = a\coth(a) - 1 - \log\frac{\sinh(a)}{a} - \frac{\beta}{2}x^2 - \beta\langle h, x\rangle + \log\frac{\sinh(b)}{b} - \frac{\beta}{2}\left(\coth(b) - \frac{1}{b}\right)^2,$$

where a *is defined by* $\coth(a) - \dfrac{1}{a} = ||x||$, b *is defined by* $\coth(b) - \dfrac{1}{b} = \dfrac{b}{\beta} - ||h||$.

Proof. From the definition of the product measure, with respect to $\mathbb{P}_n$, $\{\sigma_i\}_{i=1}^n$ are independent and identically distributed random variables, uniformly distributed on $(\mathbb{S}^2)^n$. For $t \in \mathbb{R}^3 \setminus (0, 0, 0)$, we have

$$\mathbb{E}\left(\exp\left(\langle t, \sigma_1\rangle\right)\right) = \int_{\mathbb{S}^2} \exp\left(\|t\|\langle t/\|t\|, x\rangle\right) d\mu(x). \tag{1}$$

By the symmetry, we are freely to choose our coordinate system, so we choose the Oz to lie along the vector t. Using the spherical coordinate as:

$$x_1 = \sin\varphi\cos\theta,\ x_2 = \sin\varphi\sin\theta,\ x_3 = \cos\varphi,$$

where

$$0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi,\ x = (x_1, x_2, x_3),\ t/\|t\| = (0,0,1).$$

Then the Jacobi is

$$|J| = \sin\varphi.$$

The right hand side in (1) is computed as follows:

$$\begin{aligned}\int_{\mathbb{S}^2} \exp\left(\|t\|\langle t/\|t\|, x\rangle\right) d\mu(x) &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \exp\left(\|t\|\cos\varphi\right)\sin\varphi d\varphi d\theta \\ &= \frac{1}{2}\int_0^{\pi} \exp\left(\|t\|\cos\varphi\right)\sin\varphi d\varphi \\ &= \frac{\sinh(\|t\|)}{\|t\|}.\end{aligned} \tag{2}$$

Combining (1) and (2), the cumulant generating function of σ_i is

$$c(t) = \log\mathbb{E}\left(\exp\left(\langle t, \sigma_i\rangle\right)\right) = \log\mathbb{E}\left(\exp\left(\langle t, \sigma_1\rangle\right)\right) = \log\left(\frac{\sinh(\|t\|)}{\|t\|}\right). \tag{3}$$

Since $\lim_{\|t\|\to 0}\left(\sinh(\|t\|)/\|t\|\right) = 1$, we conclude that (2) holds for all $t \in \mathbb{R}^3$. Therefore, by applying Cramér's large deviation principle for i.i.d. random variables (see, e.g., [2; p. 36]), we have S_n/n satisfies a large deviations principle with respect to the measure $\mathbb{P}_n$ with rate function

$$I(x) = \sup_{t\in\mathbb{R}^3}\left\{\langle t, x\rangle - c(t)\right\},\ x \in \mathbb{B}^2, \tag{4}$$

where $c(t)$ is the cumulant generating function of σ_1 defined as in (3). Since $I(x) = 0$ if $x = (0,0,0)$, it remains to consider the case where $x \neq (0,0,0)$. We have

$$\langle t, x\rangle - c(t) \le \|t\|.\|x\| - \log\frac{\sinh(\|t\|)}{\|t\|}.$$

Set

$$y(u) = \|x\|u - \log\frac{\sinh(u)}{u},\ u > 0.$$

We then have

$$y'(u) = \|x\| - \coth(u) + \frac{1}{u},\ y''(u) = \frac{1}{\sinh^2(u)} - \frac{1}{u^2} < 0 \text{ for all } u > 0.$$

On the other hand, $\lim_{u\to 0^+} y'(u) = \|x\| > 0$, $\lim_{u\to\infty} y'(u) = \|x\| - 1 \le 0$. These imply the equation

$$\|x\| = \coth(u) - \frac{1}{u}$$

has a unique positive solution a and $y(u)$ attains the maximum at a. It follows that

$$\begin{aligned} \sup_{t\in\mathbb{R}^3\setminus(0,0,0)} \{\langle t, x\rangle - c(t)\} &= \sup_{u>0} y(u) \\ &= \|x\|a - \log\left(\frac{\sinh(a)}{a}\right) \\ &= \left(\coth(a) - \frac{1}{a}\right) a - \log\left(\frac{\sinh(a)}{a}\right) \\ &= a\coth(a) - 1 - \log\left(\frac{\sinh(a)}{a}\right) > 0. \end{aligned} \tag{5}$$

Combining (4) and (5), we have

$$I(x) = a\coth(a) - 1 - \log\left(\frac{\sinh(a)}{a}\right), \tag{6}$$

where a is defined by $\|x\| = \coth(a) - 1/a$.

By (1.1), we can write the Hamiltonian as

$$H_n(\sigma) = -\frac{1}{2n} S_n(\sigma)^2 - \langle h, S_n(\sigma)\rangle.$$

Correspondingly, we have the Gibbs measure

$$\begin{aligned} \mathbb{P}_{n,\psi}(A) &= \frac{1}{Z}\int_A \exp\left[-\beta H_n(\sigma)\right] d\mathbb{P}_n(\sigma) \\ &= \frac{1}{Z}\int_A \exp\left[-\beta\left(-\frac{S_n^2}{2n} - \langle h, S_n\rangle\right)\right] d\mathbb{P}_n(\sigma) \\ &= \frac{1}{Z}\int_A \exp\left[-n\left(\frac{-\beta}{2}\left(\frac{S_n}{2n}\right)^2 - \beta\langle h, \frac{S_n}{n}\rangle\right)\right] d\mathbb{P}_n(\sigma) \\ &= \frac{1}{Z}\int_A \exp\left[-n\psi\left(\frac{S_n}{n}\right)\right] d\mathbb{P}_n(\sigma), \end{aligned} \tag{7}$$

where $\psi(x) = -\frac{\beta}{2}x^2 - \beta\langle h, x\rangle$. From (4), (5) and (7), by applying Proposition 1.3, we conclude that S_n/n satisfies a large deviation principle with respect to the Gibbs measures $\mathbb{P}_{n,\psi}$ with rate function:

$$\begin{aligned} I_\psi(x) &= I(x) + \psi(x) - \inf_{y\in\mathbb{B}^2}\{I(y) + \psi(y)\} \\ &= a\coth(a) - 1 - \log\left(\frac{\sinh(a)}{a}\right) - \frac{\beta}{2}x^2 - \beta\langle h, x\rangle - \inf_{y\in B^2}\{I(y) + \psi(y)\},\ x \in \mathbb{B}^2, \end{aligned} \tag{8}$$

where a is defined by $\|x\| = \coth(a) - 1/a$. Now, we will compute

$$\inf_{y\in B^2} \{I(y) - \psi(y)\}.$$

By (6) and the fact that $|\langle h, x\rangle| \le \|h\|\|x\|$, we have

$$\begin{aligned}\inf_{y\in B^2} \{I(y) + \psi(y)\} &= \inf_{a\ge 0} \left\{ a\coth(a) - 1 - \log\frac{\sinh(a)}{a} - \frac{\beta}{2}\left(\coth(a) - \frac{1}{a}\right)^2 - \beta\|h\|\left(\coth(a) - \frac{1}{a}\right)\right\} \\ &= \inf_{a\ge 0} \left\{ \left(\coth(a) - \frac{1}{a}\right)(a - \beta\|h\|) - \log\frac{\sinh(a)}{a} - \frac{\beta}{2}\left(\coth(a) - \frac{1}{a}\right)^2\right\}.\end{aligned} \tag{9}$$

Let

$$f(u) = \left(\coth(u) - \frac{1}{u}\right)(u - \beta\|h\|) - \log\frac{\sinh(u)}{u} - \frac{\beta}{2}\left(\coth(u) - \frac{1}{u}\right)^2, \ u > 0.$$

We have

$$f'(u) = \left(\frac{1}{u^2} - \frac{1}{\sinh^2(u)}\right)\left(u - \beta\left(\coth(u) - \frac{1}{u}\right) - \beta\|h\|\right). \tag{10}$$

Let

$$g(u) = u - \beta\left(\coth(u) - \frac{1}{u}\right) - \beta||h||, \ u > 0. \tag{11}$$

Then $g(u) = 0$ if only if

$$\begin{aligned}\beta &= \frac{u}{\coth(u) - 1/u + ||h||} \\ &= \frac{u^2}{u\coth(u) + ||h||u - 1} \\ &:= k(u).\end{aligned} \tag{12}$$

We have

$$k^{'}(u) = \frac{u^2\left(\coth(u) + ||h|| - 2/u + u/\sinh^2(u)\right)}{(u\coth(u) + ||h||u - 1)^2}.$$

By elementary calculations, we can show that (see [6; p85])

$$\coth(u) - \frac{2}{u} + \frac{u}{\sinh^2 u} > 0 \text{ for all } u > 0.$$

It implies that the function $k(u)$ is strictly increasing on $(0, \infty)$. Moreover, expanding the function $\coth(u)$ in Taylor series, we have $\lim_{u\to 0^+} k(u) = 0$, $\lim_{u\to\infty} k(u) = \infty$. This and

(12) imply that equation $k(u) = \beta$ has a unique positive solution b, and therefore, from the definition of $g(u)$ in (11), we have

$$g(u) < 0 \text{ for all } u \in (0, b),\ \ g(u) > 0 \text{ for all } u \in (b, \infty). \tag{13}$$

Since $\lim_{u\to 0^+} f(u) = 0$ and $1/a^2 - 1/\sinh^2(a) > 0$ for all $a > 0$, combining (10), (11) and (13), we obtain

$$\inf_{a>0} \left\{ \left(\coth(a) - \frac{1}{a}\right)(a - \beta\|h\|) - \log\frac{\sinh(a)}{a} - \frac{\beta}{2}\left(\coth(a) - \frac{1}{a}\right)^2 \right\} = f(b) < 0. \tag{14}$$

Combining (8), (9) and (14), we have for all $x \in \mathbb{B}^2$,

$$\begin{aligned} I_\psi(x) &= a\coth(a) - 1 - \log\frac{\sinh(a)}{a} - \frac{\beta}{2}x^2 - \beta\langle h, x\rangle - f(b) \\ &= a\coth(a) - 1 - \log\frac{\sinh(a)}{a} - \frac{\beta}{2}x^2 - \beta\langle h, x\rangle + \log\frac{\sinh(b)}{b} - \frac{\beta}{2}\left(\coth(b) - \frac{1}{b}\right)^2, \end{aligned}$$

where a is defined by $\coth(a) - \dfrac{1}{a} = ||x||$, b is defined by $\coth(b) - \dfrac{1}{b} = \dfrac{b}{\beta} - ||h||$. This proves the theorem. □

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TÓM TẮT

NGUYÊN LÝ ĐỘ LỆCH LỚN CHO MÔ HÌNH TRƯỜNG TRUNG BÌNH HEISENBERG VỚI TỪ TRƯỜNG NGOÀI

Trong bài báo này, chúng tôi xét mô hình trường trung bình Heisenberg với từ trường ngoài tất định. Chúng tôi chứng minh nguyên lý độ lệch lớn cho S_n/n theo độ đo Gibbs, trong đó S_n là tổng spin. Đặc biệt, chúng tôi thu được biểu thức tường minh cho hàm tốc độ.